TABLE 1. COMPARISON OF DISTRIBUTED SUBOPTIMAL CONTROL TECHNIQUES

Method	Lyapunov		Instantaneous minimization criterion (1)
	LGL	*DSL*	
key function	$V(t) = \langle u,Qu \rangle$		$\phi(u,v,t)$
objective	minimize $\dot{V}(t_k) = \Lambda_k, \forall\ k$		minimize $\phi(t), \forall\ t$
control law	$\nabla\Lambda_k(v_k) \equiv 0$		$\eta\phi_{vk+1} + \phi_{uk+1} = 0$
spec. c.l.	$f_v(u_k)Qu_k \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \Rightarrow v_k(x) = \begin{matrix} v^+ \\ 0 \\ v_+ \end{matrix}$		$v_{k+1}(x) = -\dfrac{\alpha}{\eta} F(u_x,v_{k+1})$
applicability:		v_k min Λ_k	
$v = v(t)$	yes	yes	yes
$v = v\ (x,t)$	yes	no	yes
hyperbolic	yes	yes	yes
parabolic	yes	yes	yes
linear state	yes	yes	yes
nonlinear state	yes	yes	yes
linear control	yes	yes	yes
nonlinear control	no	yes	yes
adjustable parameters	Q	Q	η,N

The table also contains information as to the range of applicability of each method.

All these methods share the advantage of producing a feedback control using very little computational effort. Of the three, instantaneous minimization has the widest range of applicability. The DSL method is restricted to systems in which the control is a function of time only, while the LGL method is restricted to systems linear in the control.

One definite advantage held by the instantaneous minimization method is the fact that the actual performance functional kernel is used, whereas in a Lyapunov scheme the Lyapunov functional may bear little relation to the actual performance functional, thus degrading the quality of the resulting suboptimal control.

Part II. Discussion of Test Systems and Numerical Results

The suboptimal control algorithms developed in Part I are tested on four example problems to obtain some indication of their performance. Results are presented and compared with results obtained from the application of classical open-loop optimal control synthesis algorithms to the same test problems.

The suboptimal feedback control algorithms are found to give acceptable performance on both linear and nonlinear systems at the expense of very little computational effort.

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SCOPE

The performance of the suboptimal control algorithms developed in Part I of this work is to be evaluated by direct comparison of the effect of suboptimal control to the effect of classically optimal control upon four specific test problems.

The basic test systems are a linear parabolic system, a linear first-order hyperbolic system, a linear parabolic system with variable coefficients, and a nonlinear parabolic system.

These four classes encompass a great number of the distributed systems encountered in engineering practice and are thus sufficient to provide the indications of performance we desire.

It should be noted that a rigorously optimal control for a linear system with a quadratic performance functional may be obtained in feedback form but only at the cost of extensive computation. No method for the computation of

an optimal feedback control for nonlinear systems, or linear systems with a nonquadratic performance functional, is yet known. For this reason, the generation of near-optimal feedback controls for all types of distributed systems would be of great potential use, particularly if little computational effort is required.

CONCLUSIONS AND SIGNIFICANCE

From examination of the numerical and graphical results presented here, one may readily deduce that suboptimal control applied to certain distributed systems yields very acceptable results with little expenditure of computational effort. Furthermore, all the advantages of a feedback control loop are gained. Small parameter variations have no adverse effect on these suboptimal control schemes, which are applicable to linear and nonlinear systems alike.

Included in the instantaneous minimization and multilevel bang-bang schemes are free parameters which may be adjusted in order to obtain the highest quality of suboptimal control for the particular system under consideration. This fact lends considerable flexibility to these control schemes in practical application.

The two Lyapunov methods provide acceptable control with the added guarantee of stability for the controlled system. The linear gradient Lyapunov (LGL) method is suited to systems with the control as a function of space and time, and which are linear in the control. The direct search Lyapunov (DSL) method will accept systems nonlinear in the control term, but with the control as a function of time only.

These suboptimal control methods, all of which may be implemented on the smallest of process control computers, may be substantially improved by further investigation of their properties. The choice of Q, the operator in the Lyapunov functional, may greatly affect the quality of control obtainable via Lyapunov methods. Chant and Luus (1968) have performed investigations in this area on lumped-parameter systems.

In multilevel schemes, the choice of intermediate values u^+ and v^+ affects the quality of control. In practical applications the control loop could be "tuned" by selection of these parameters.

Computation time for the suboptimal feedback techniques was small enough as to allow the completion of a typical run in less than one second, including simulation of the response of the controlled system. Synthesis of open-loop optimal controls required from between 1 to 9 sec., depending upon the choice of synthesis technique and test system. All runs were performed on an IBM 360/91 computer. These results indicate that the suboptimal feedback control techniques discussed here are suitable for implementation on the vast majority of chemical processes whose characteristic times exceed a few hundredths of a second. Detailed compilation of computing times may be found in Vermeychuk (1972).

In summary, suboptimal feedback control for distributed systems of the type described here offers the advantages of simplicity and flexibility, coupled with good performance, as is demonstrated.

DESCRIPTION OF TEST SYSTEMS

Here we will present four example problems used to test the suboptimal control algorithms presented in Part I. All of these examples have one state and one control variable, are written in one spatial dimension, and are volume controlled.

The examples explained in turn are the linear parabolic system, linear first-order hyperbolic system, linear parabolic system with variable coefficients, and the nonlinear parabolic system.

An elliptic case was not included because of the rare necessity in chemical engineering for the control (as opposed to analysis) of phenomena governed by an elliptic PDE, such as steady state heat conduction. A second-order hyperbolic or wave equation was not included due to the rare occurrence of phenomena described by this equation in chemical engineering. This equation should not be confused with the first-order hyperbolic equation of chromatography and the like, in which only first-order time derivatives appear.

Linear Parabolic System

System: $u_t(x,t) = u_{xx}(x,t) + v(x,t)$

$$x \epsilon (0,1); \; t \epsilon [t_0, t_f]$$

$$u(x,t_0) = u_0(x) = 1 - 4(x - 1/2)^2$$

$$u(0,t) = u(1,t) = 0$$

Performance functional:

$$J = \int_{t_0}^{t_f} \left[\int_0^1 \int_0^1 u(x,t) Q(x,x') u(x',t)\, dx dx' + \gamma \int_0^1 v^2(x,t)\, dx \right] dt$$

Note that the system will tend to its own desired state $u_D \equiv 0$ at equilibrium if no control action is applied. Thus, the control must accelerate the state of the system toward the origin. At times, we considered the system with the alternative boundary conditions

$$u(x,t_0) = u_0(x) = -1.0$$

$$u(0,t) = u(1,t) = -1.0$$

which render the system at equilibrium at the initial time. The control must then drive the system to the origin while minimizing J.

This example can serve as a basis for the study of simple diffusion and heat conduction situations.

Linear Hyperbolic System—Flow-through Heating

Consider the system depicted in Figure 1.

The sheet, entering the furnace at velocity α with temperature u_0, is to be heated to an exit temperature T. The temperature inside the furnace is $v(t)$, and the heat transfer between the sheet and the furnace atmosphere is characterized by the constant σ.

The Biot number of the sheet is defined as

$$Bi = \sigma b / 2k$$

where b is the thickness of the sheet and k is the thermal conductivity. When $Bi \leqq 0.25$, variations of u in the x_2 direction may be ignored. If we also neglect axial heat conduction, we obtain:

System:
$$u_t(x,t) + \alpha u_x(x,t) = \sigma(v(t) - u(x,t))$$
$$x \epsilon (0,1); \; t \epsilon [t_0, t_f]$$
$$u(0,t) = u_0 = \text{constant}, \; 0 < u_0 < T$$
$$u(x,t_0) = u_{ss}(x) = u_0 e^{-\nu x} + \nu v(0) e^{\nu x}; \quad \nu = \sigma/\alpha$$

Performance functional:

$$J = \int_{t_0}^{t_f} (u(1,t) - T)^2 \, dt$$

T is the desired exit temperature, and $u_{ss}(x)$ is obtained by setting $u_t = 0$ and solving the system equation by Laplace transforms at $t = 0$.

Parabolic System with Variable Coefficient

System:
$$u_t(x,t) = u_{xx}(x,t) + \psi(x) v(t)$$
$$x \epsilon (-1,1); \; t \epsilon [t_0, t_f]$$
$$u(x,t_0) = u_0(x) = -c$$
$$\nabla u(x,t) + N u(x,t) = 0, \; x = \pm 1$$
N = Nusselt number.

Performance functional:

$$J = \int_{t_0}^{t_f} \left[\int_0^1 \int_0^1 uQu' \, dx dx' + \gamma \int_0^1 v^2 \, dx \right] dt$$

$$\psi(x) = (\bar{\rho} \alpha^2 / \rho C_P) \left[\frac{A\phi + B\theta}{A^2 + B^2} \right]^2$$
$$A = \cos z \cosh z$$
$$B = \sin z \sinh z$$
$$\phi = \cos w \sinh w - \sin w \cosh w$$
$$\theta = \sin w \cosh w + \cos w \sinh w$$
$$w = \alpha x$$
$$z = \frac{1}{2} w$$
$$\alpha = (\omega \mu_r \mu_0 / 2 \bar{\rho})^{1/2}$$

This example is taken from the heating of a semi-infinite slab of conducting material by electromagnetic induction. A full derivation of the model equations appears in Vermeychuk (1972) using the treatment of Sundberg (1965).

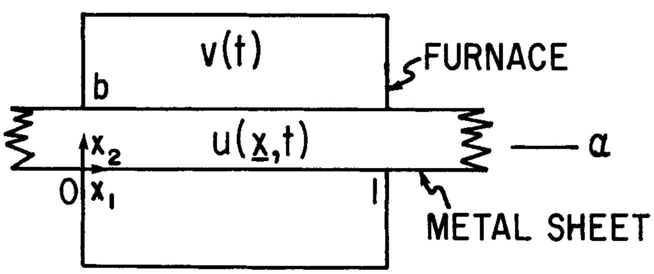

Fig. 1. Flow-through heating system in example "B."

Nonlinear Parabolic System

This example is based on the case of diffusion with concentration-dependent diffusivity discussed by Crank (1956).

System:
$$u_t(x,t) = k(u(x,t)) u_{xx}(x,t) + k_u(u(x,t)) \, u_x^2(x,t) + v(x,t)$$
$$x \epsilon (0,1); \; t \epsilon [t_0, t_f]$$
$$u(x,t_0) = u_0(x) = -1.0$$
$$u(0,t) = u(1,t) = -1.0$$

Performance functional:

$$J = \int_{t_0}^{t_f} \left[\int_0^1 \int_0^1 uQu' \, dx dx' + \gamma \int_0^1 v^2 \, dx \right] dt$$

Note that this example is fully nonlinear in the state. We used

$$k(u) = a + bu(x,t)$$
$$a = 1.00$$
$$b = 0.01$$

for our test runs. Numerical solution of the nonlinear state and adjoint equations required a new numerical method which is presented by Vermeychuk (1972).

NUMERICAL RESULTS

In this section, we present numerical results obtained on the test systems with the various suboptimal control techniques in Part I and compare the suboptimal control policies with optimal control policies generated by the gradient (GRAD) method of Seinfeld and Lapidus (1968) and original procedures based on the second variation (DSV) and contraction mapping (CONMAP) presented by Vermeychuk.

Hyperbolic Multilevel Bang-Bang (MLBB) Control

System: Example "B"
$u_0 = 1.00$ $\quad T = 3.00$ $\quad v_0 = 1.0$

Optimal control: Gradient
$a^2 = 0.09$ $\quad \epsilon = 0.0005$

Suboptimal control: MLBB $u^{\mp} = 2.5$
$v_+ = 0.0$ $\quad v^{\mp} = 10.0$ $\quad v^+ = 30.0$

Results:	no control	optimal	suboptimal
Final P.I.	4.76977	0.16977	0.43561

Figures 2 and 3 represent the suboptimal control and the outlet state of the system as functions of time, respectively. These graphs may be directly compared with Figures 4 and 5, depicting the control and outlet state of the same system under optimal control and outlet state of the same system under optimal control produced by the gradient method. Note that in the suboptimally controlled case with $v^+ = 30.0$ the outlet state requires a longer time to reach the desired value and exhibits no overshoot, as in the optimally controlled case.

Faster attainment of the desired state under suboptimal control could be accomplished by increasing the value of v^+. The suboptimally controlled case presented here, however, exhibits very acceptable performance when compared with the optimally controlled case, in terms of attainment of the desired state and value of the performance index.

The quality of these results indicates that this form of suboptimal control may be considered as the control technique of choice in certain applications, due to the simplicity of the control algorithm and to the natural advantages of feedback operation.

Suboptimal Control with the Lyapunov Functional

System: Example "A"

$$J = \mu||u||^2 + \gamma||v||^2, \quad \mu = 1.0 \quad \gamma = 0.10$$

$$u_0(x) = -1.0 \quad u(0, t) = u(1, t) = 1.0$$

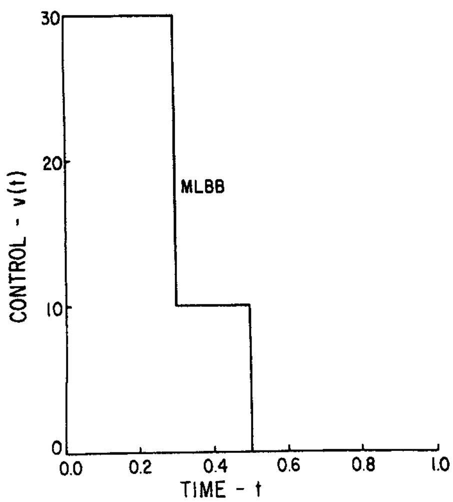

Fig. 2. Example "B": suboptimal control vs. time.

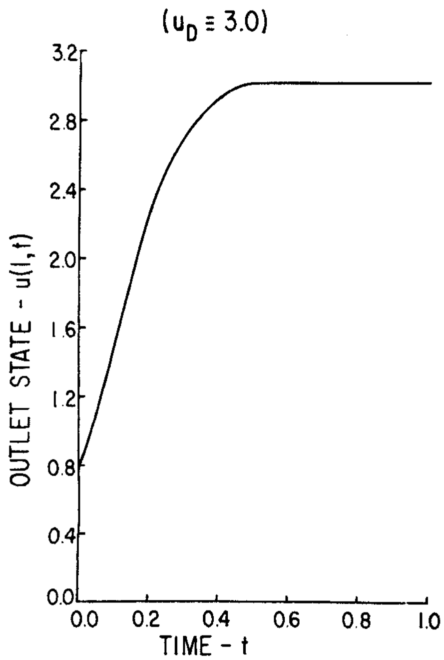

Fig. 3. Example "B": outlet state vs. time under suboptimal control ($u_D = 3.0$).

Optimal control: Gradient; $a^2 = 0.0001$, $\epsilon = 0.0005$
Contraction mapping → *TPBVP*
Suboptimal control:

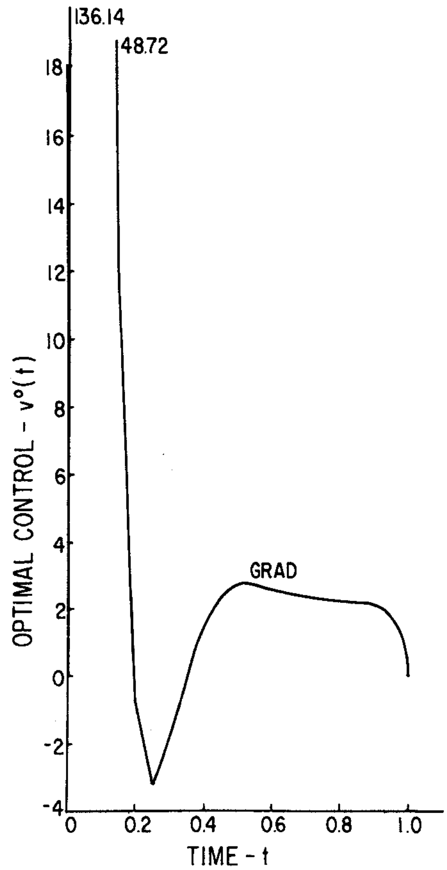

Fig. 4. Example "B": open-loop optimal control vs. time.

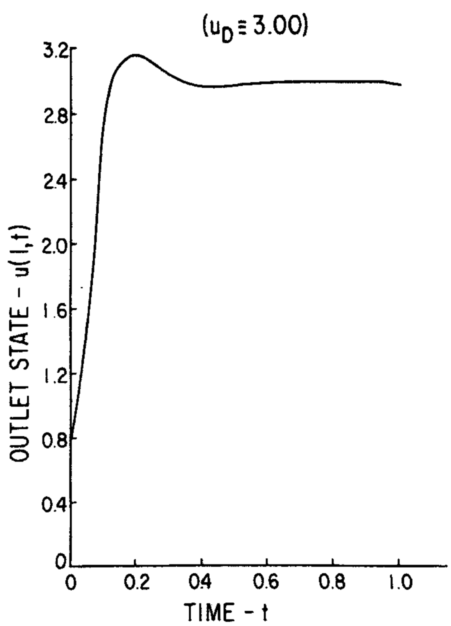

Fig. 5. Example "B": outlet state vs. time under optimal control ($u_D = 3.0$)

LGL; $v^+ = 0.50$, $v_+ = 0.00$

DSL; $v^+ = 0.50$, $v_+ = 0.00$, steps of 0.01

Results:

Method	Final P. I.
no control	0.90222
gradient	0.85389
contraction mapping	0.84835
LGL	0.86077
DSL	0.86077

Figures 6 and 7 illustrate the centerline optimal control vs. time and the final state for example A under suboptimal control of the Lyapunov type, as well as for instantaneous kernel minimization (IKM) suboptimal control. The IKM suboptimal control will be discussed subsequently.

The identical control provided by both of the Lyapunov methods is a consequence of the fact that the test system is linear in both the state and the control. Lyapunov sub-

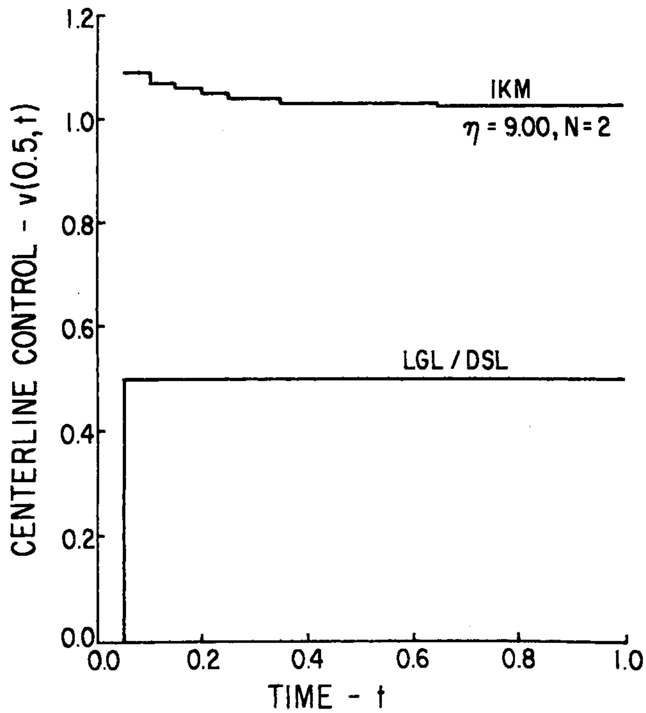

Fig. 6. Example "A": suboptimal control at centerline vs. time.

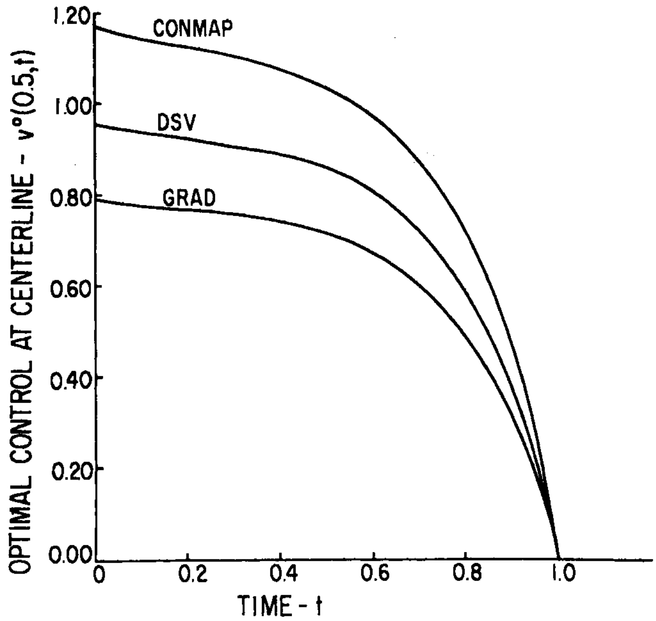

Fig. 8. Example "A": open-loop optimal controls at centerline vs. time.

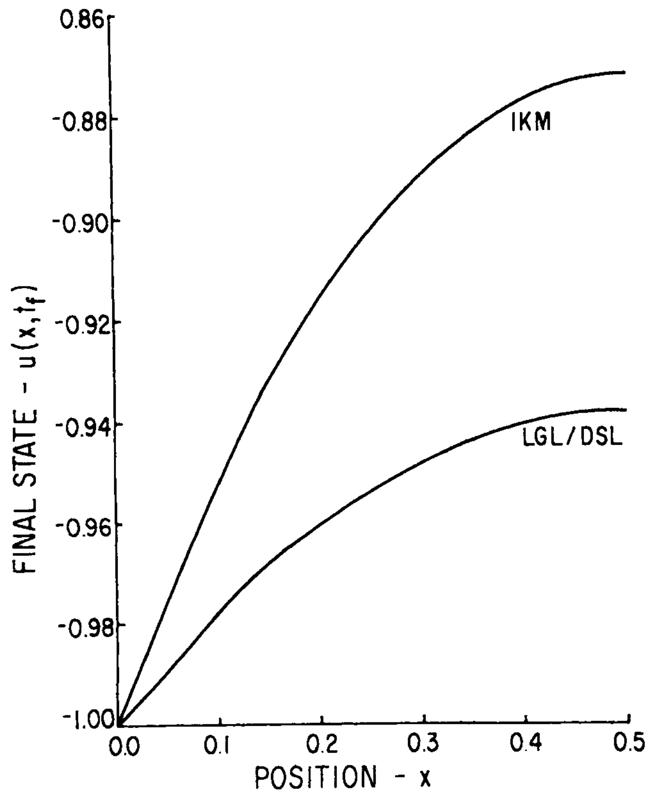

Fig. 7. Example "A": final state vs. position under suboptimal control ($u_D = 0.0$).

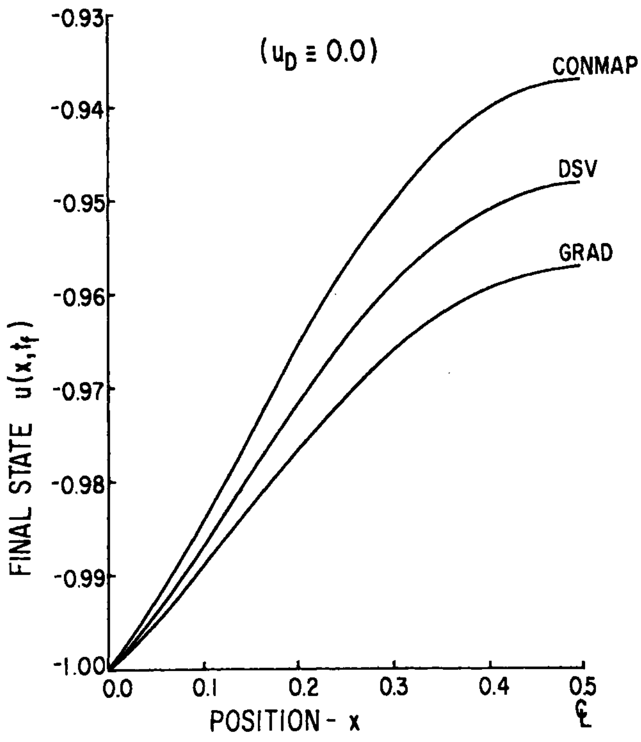

Fig. 9. Example "A": final state vs. position under optimal control ($u_D = 0.0$)

optimal control provides an acceptable value of the performance index, and upon comparing Figures 6, 7, 8, and 9 it can be seen that the Lyapunov control drives the state of the system closer to the desired value than does the optimal control.

System: Example "C"

$$J = \mu||u - u_D||^2 + \gamma||v||^2, \; \mu = 1.0 \quad \gamma = 0.10$$

$$u_0 \equiv 0.0 \qquad u_D \equiv 1.0$$

Optimal control: Gradient; $a^2 = 0.0025$, $\epsilon = 0.0005$
Suboptimal control: LGL; $v^+ = 2.50$ $v_+ = 0.00$

Results:	No control	Gradient	LGL
Final P. I.	1.86889	1.64795	1.89725

Figures 10 and 11 represent the control vs. time and the final state for this example under Lyapunov and IKM

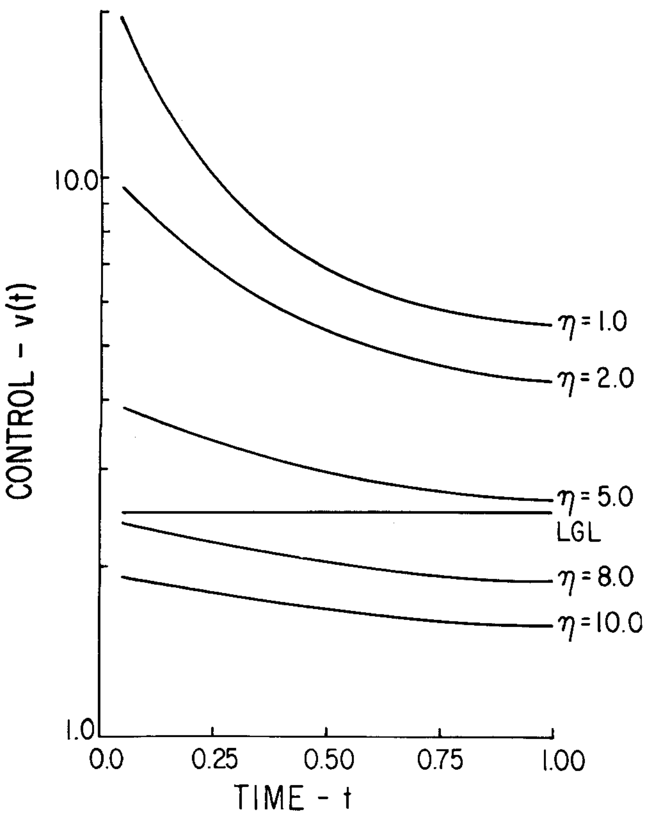

Fig. 10. Example "C": suboptimal control vs. time, LGL and IKM.

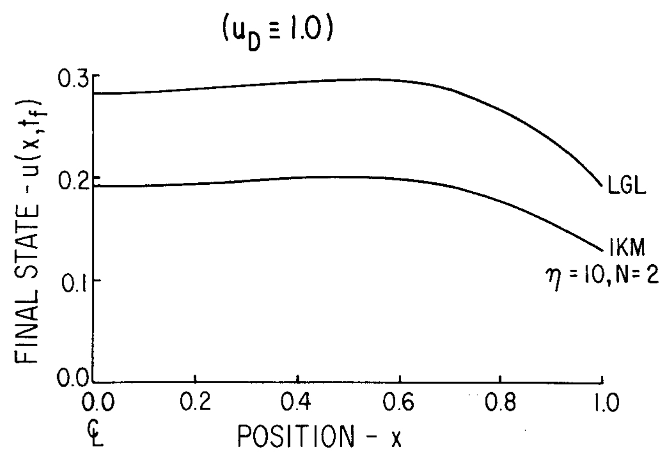

Fig. 11. Example "C": final state vs. position under suboptimal control ($u_D = 1.0$).

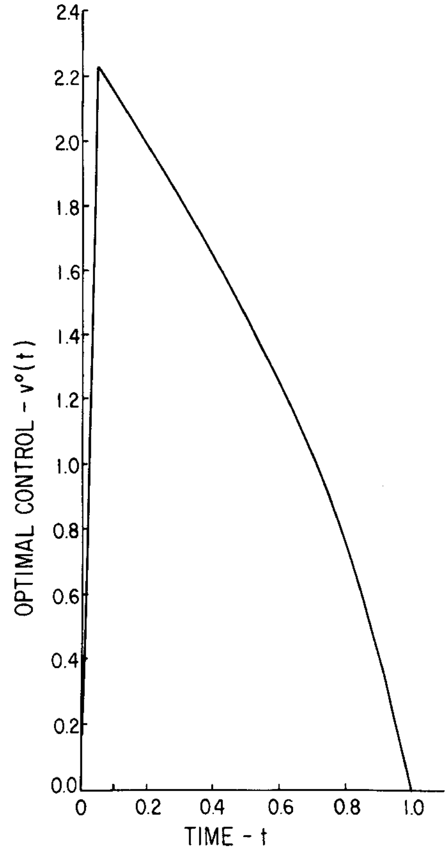

Fig. 12. Example 'C": open-loop optimal control vs. time.

suboptimal control. It can be seen that the LGL method provided too much control effort for the system. It is true that the final state under LGL control is nearer the desired state than with any form of open-loop optimal control (compare Figures 11 and 13), but the additional control effort required to bring about this effect (Figures 10 and 12) was sufficient to yield a performance index for LGL control which exceeds the performance index obtained for no control at all. An improvement in the quality of LGL control for this system could be realized by reducing the value of v^+.

System: Example "D"

$$J = \mu||u||^2 + \gamma||v||^2, \; \mu = 1.0 \quad \gamma = 0.10$$

$$u_D \equiv -1.0 \qquad u_D \equiv 0.0$$

Optimal control: Gradient; $a^2 = 0.0001$, $\epsilon = 0.0005$
Contraction mapping → *TPBVP*
Suboptimal control:

LGL; $v^+ = 0.50$ $v_+ = 0.00$

DSL; $v^+ = 0.50$ $v_+ = 1.00$ steps of 0.01

Results:	Method	Final P. I.
	no control	0.90222
	gradient	0.85335
	contraction mapping	0.84826
	LGL	0.86071
	DSL	0.86071

Figures 14 and 15 display the control provided for the nonlinear parabolic system by both LGL and DSL methods and the final state realized under this form of suboptimal control. Also shown on these graphs are results of

IKM suboptimal control, which will be discussed shortly. Figures 16 and 17 illustrate the optimal control.

The control obtained for this example was identical with both LGL and DSL methods since the system is linear in the control, even though it is nonlinear in the state. This control yields acceptable performance with the advantages of feedback operation.

Parabolic Suboptimal Control with Instantaneous Kernel Minimization

Before presenting the results of the application of this (IKM) method to our test systems, we will display some numerical results which indicate the effect of variations in N and η, the two parameters in the control algorithm, on the quality of the suboptimal control obtained. A modified version of example "A" was used to obtain these results, which are presented in tabular form, since the scale of the control ordinates involved was not conducive to adequate representation in graphical form. Table 1, then, is a record of centerline control vs. time for the IKM method operating on the modified version of example "A" to follow, under different values of η and N.

The special test system used to produce the results in the table is:

Example "A"

$$u_D = 0.0 \quad u_{eq} = 0.0 \quad u_0 = 1 - 4(x - 0.5)^2$$

$$u(0, t) = u(1, t) = 0.0$$

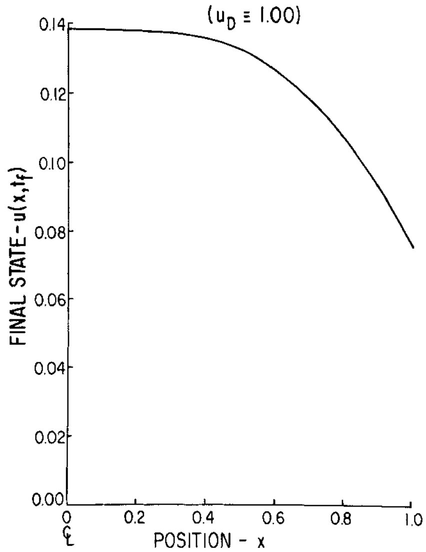

Fig. 13. Example "C": final state vs. position under optimal control ($u_D = 1.0$).

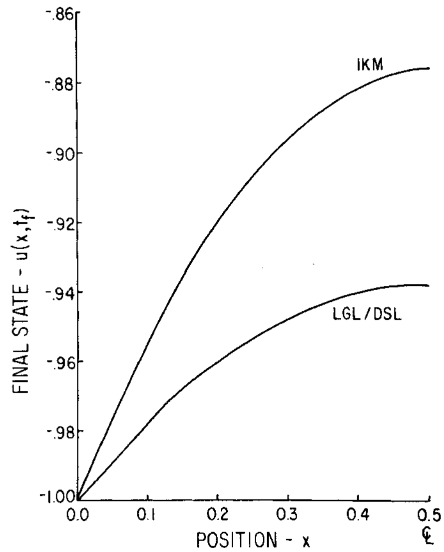

Fig. 15. Example "D": final state vs. position under suboptimal control.

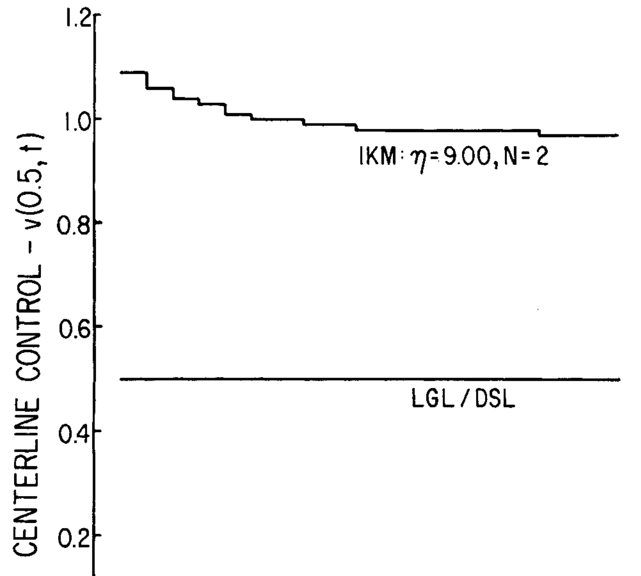

Fig. 14. Example "D": suboptimal control at centerline vs. time.

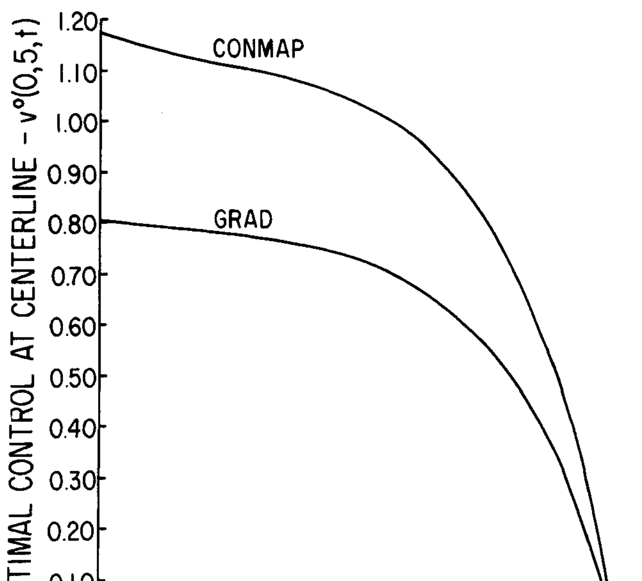

Fig. 16. Example "D": open-loop optimal control at centerline vs. time.

$$J = \mu||u||^2 + \gamma||v||^2 \quad \mu = \gamma = 1.00$$

With no control, the performance index obtained is 0.07373. With optimal control by the gradient method, a performance index of 0.01055 is obtained.

Following Table 1 descriptive comments on the behavior of the centerline control vs. time will be given, along with the resulting performance index, if applicable.

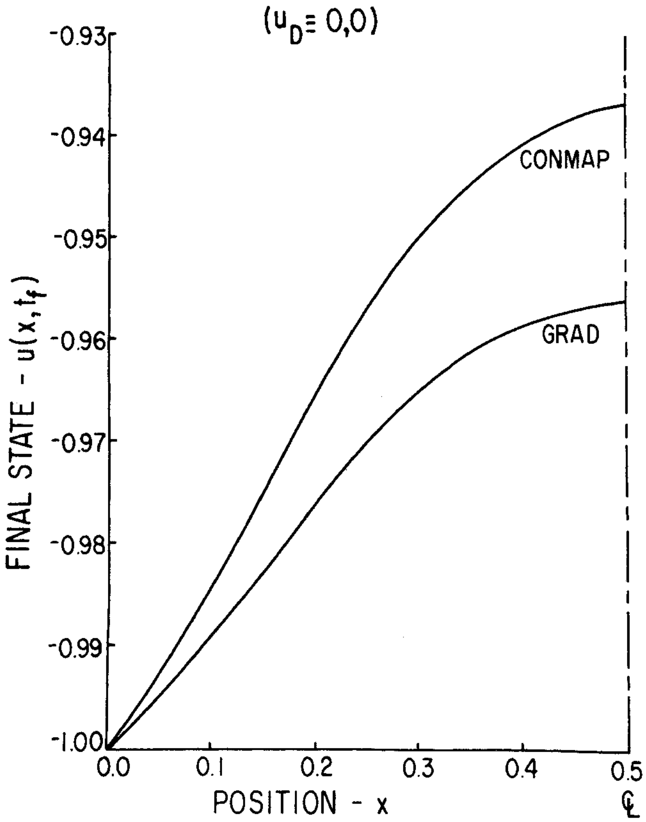

Fig. 17. Example "D": final state vs. position under optimal control ($u_D = 0.0$).

Inspection of Table 1 reveals the following:

	N		behavior of $v(t)$	J
1.00	0	guess only	oscillatory divergence	
4.00	1		oscillatory divergence	
5.00	1		osc. div., smaller envelope	
4.00	2		osc. div., decreasing freq.	
5.00	2		small oscillation	0.49476
5.00	3		osc. adds harmonics	
7.00	2		small oscillation	0.02023
9.00	2		gradual decrease	0.01595

It can be seen that proper selection of N and η is very important in achieving the type of control desired. The case with $N = 2$ and $\eta = 9.00$ gave good results in this initial test, so we chose those values for use in the control tests on examples "A" and "D" for comparison with other control techniques. As for the effects of η and N, the parameter η seems to affect the stability of the controlled system as would the gain of a proportional controller. Increasing the value of η corresponds to decreasing the proportional gain. The parameter N, the number of iterations of (39) in Part I for each time interval, seems to have a preferred value of 2 in this particular system.

The IKM method on this test system yields very good performance when compared with the open-loop gradient method.

We now present specific results obtained upon application of the IKM method to examples "A," "D," and "C."

System: Example "A"

$$J = \mu||u||^2 + \gamma||v||^2\,, \; \mu = 1.0, \quad \gamma = 0.10$$

$$u_0 = -1.00 \qquad u_0 = 0.0$$

Optimal control: Gradient
Contraction mapping → *TPBVP*
Suboptimal control: IKM; $N = 2$, $\eta = 9.00$

Results:	Method	P. I.
	no control	0.90222
	gradient	0.85389

TABLE 1. IKM SUBOPTIMAL CONTROL, $v(0.5,t)$ VS. t ON EXAMPLE "A"

	t = 1.00 N = 0	4.00 1	5.00 1	4.00 2	5.00 2	5.00 3	7.00 2	9.00 2
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.05	0.57143	0.03599	0.05658	0.20565	0.12735	0.02681	0.07716	0.06002
0.10	−1.53890	0.00483	0.01536	0.28855	0.14910	−0.01231	0.07376	0.05177
0.15	0.93326	0.03864	0.04335	0.35028	0.11910	0.05596	0.04921	0.03529
0.20	−7.09563	−0.02340	0.00192	0.25217	0.03810	−0.03476	0.02334	0.02189
0.25	79.20143	0.08250	0.04183	0.07232	−0.03967	0.05013	0.00783	0.01409
0.30	OFLOW*	−0.13368	−0.01762	−0.29210	−0.10365	−0.00718	−0.00147	0.00846
0.35		0.31079	0.05648	−0.65057	−0.11173	−0.00995	−0.00283	0.00596
0.40		−0.63383	−0.05416	−1.01399	−0.08326	0.06854	−0.00281	0.00378
0.45		1.38332	0.10200	−1.07773	−0.01511	−0.09226	−0.00041	0.00304
0.50		−2.95766	−0.13442	−0.84426	0.04632	0.12785	0.00025	0.00201
0.55		6.41595	0.22145	−0.06318	0.09330	−0.01894	0.00153	0.00126
0.60		−13.90365	−0.33272	1.02986	0.04315	0.07681	0.00111	0.00116
0.65		30.27613	0.53930	2.38946	0.06399	0.01320	0.00134	0.00106
0.70		−66.05080	−0.86087	3.37651	0.00529	−0.10852	0.00059	0.00068
0.75		144.46368	1.41382	3.67905	−0.04441	0.22417	0.00064	0.00064
0.80		OFLOW*	−2.32731	2.55076	−0.07928	−0.29408	0.00011	0.00029
0.85		694.52733	3.87652	0.01478	−0.07447	0.31678	0.00025	0.00038
0.90		OFLOW*	−6.48157	−3.97816	−0.04668	−0.23667	−0.00004	0.00022
0.95			10.89170	−8.24884	−0.00288	0.07221	0.00014	0.00023
1.00			−18.34281	−11.65266	0.04231	0.18391	−0.00004	0.00012

* Indicates number of digits exceeded print format.

contraction mapping	0.84835
IKM	0.86563
LGL/DSL	0.86077

Figures 6 and 7 show suboptimal control vs. time and final state for the IKM method applied to this problem. The performance index obtained compares favorably with that obtained when Lyapunov suboptimal control techniques were employed. Further improvement in the IKM control might be had through precise adjustment of the parameter η.

System: Example "D"
Optimal control: Gradient
Contraction mapping → *TPBVP*
Suboptimal control: IKM; $N = 2$, $\eta = 9.00$

Results:	Method	P. I.
	no control	0.90222
	gradient	0.85335
	contraction mapping	0.84826
	IKM	0.86448
	LGL/DSL	0.86071

Figures 14 and 15 show the results obtained with IKM suboptimal control applied to the nonlinear parabolic system. As with the linear system "A," the IKM produced results comparable to those obtained with the Lyapunov methods, subject to improvement through adjustment of the free parameters in the control algorithm.

In the next example, the gains to be realized through proper adjustment of the parameter η are more fully explored.

System: Example "C"

Results:	Method			P. I.
	no control			1.86889
	gradient			1.64795
	LGL			1.89725
	IKM;	$N = 2$,	$\eta =$ 1.0	8.37780
			2.0	4.13925
			5.0	2.07563
			8.0	1.78953
			10.0	1.73758

Figure 10 shows the control vs. time behavior for the IKM method with these various values of the parameter η, while Figure 11 depicts the final state obtained with the best of the IKM variations. It is evident that proper selection of η can lead to a suboptimal control producing near-optimal results in this system, which is parabolic with a position dependent control coefficient. Feedback operation is provided, and the same basic IKM algorithm is utilized.

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NOTATION

G = Green's operator
G^* = adjoint Green's operator
J = performance index
Q = operator (matrix or integral kernel)
R = operator (matrix or integral kernel)
T = desired temperature—example "B"
$V(t)$ = Lyapunov functional
a^2 = step size in gradient method
$g(x, t; \xi, \tau)$ = Green's function
t = time
$u(x, t)$ = state
u_D = desired state
u_{eq} = equilibrium state
u_0 = initial state
u_{ss} = steady state value of state—example "B"
u^+ = intermediate value of state in MLBB method
v = control
v^+ = intermediate value of control in MLBB method
x = position

Greek Letters

$\Lambda(t)$ = time derivative of the Lyapunov functional
Ω = spatial domain
α = constant
β = constant
γ = constant
δ = variation
ϵ = preselected tolerance
η = constant
μ = constant
μ_0 = magnetic constant $4\pi \times 10^{-7}$ Henry/m
μ_r = relative permeability
ρ = density
$\bar{\rho}$ = resistivity
σ = heat transfer coefficient—example "B"
ϕ = temporal kernel of performance functional
$\psi(x)$ = specific function of position in example "C"
ω = angular frequency

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